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Dirac Operators on Coset Spaces

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Abstract

The Dirac operator for a manifold Q, and its chirality operator when Q is even dimensional, have a central role in noncommutative geometry. We systematically develop the theory of this operator when $Q = G/H$, where G and H are compact connected Lie groups and G is simple. An elementary discussion of the differential geometric and bundle theoretic aspects of G/H, including its projective modules and complex, Kähler and Riemannian structures, is presented for this purpose. An attractive feature of our approach is that it transparently shows obstructions to spin- and spin$_c$-structures. When a manifold is spin$_c$ and not spin, $U(1)$ gauge fields have to be introduced in a particular way to define spinors [1, 2]. Likewise, for manifolds like $SU(3)/SO(3)$, which are not even spin$_c$, we show that $SU(2)$ and higher rank gauge fields have to be introduced to define spinors. This result has potential consequences for string theories if such manifolds occur as D-branes. The spectra and eigenstates of the Dirac operator on spheres $S^n = SO(n+1)/SO(n)$, invariant under $SO(n+1)$, are explicitly found. Aspects of our work overlap with the earlier research of Cahen et al. [2].

1 Introduction

When a group G acts transitively on a manifold Q with stability group H at a point p, we can identify Q with the coset space G/H. Such spaces are important in the description of Goldstone modes created by the spontaneous breakdown of G to H. Models of spacetime such as the Minkowski spacetime $M^{3,1}$ or its compact Euclidean version S^4 are also of this sort. The group G in these cases is the Poincarè group and $SO(5)$ respectively, while H is the Lorentz group and $SO(4)$ respectively. In addition, coset spaces like $\mathbb{C}P^N$ and S^N have begun to proliferate as $D-$branes in string and boundary conformal field theories.

The Dirac operator for a manifold Q, and its chirality operator when Q is even–dimensional, have a central role in noncommutative geometry. That is a good motivation for their study. This work focuses on this enterprise when Q is a coset space. In addition, in a subsequent paper, we shall develop fuzzy versions of certain coset spaces and their Dirac and chirality operators, primarily as a device to regularize quantum field theories thereon, and what we do here is also a preparation for it.

We assume throughout that G is a simple compact connected Lie group and H is a compact connected group. Without loss of generality we assume also that G is simply connected. These restrictions on G and H can be relaxed somewhat, G can be semi–simple for instance, and certain noncompact Lie groups G too seem approachable by our methods.

Not all G/H admit a spin–, or even a spin$_c$–structure [3]. One attractive aspect of our approach is that obstructions to spin– and spin$_c$–structure show up transparently and we can also easily see when and how we can overcome them using suitable generalized spin–structures, spin$_K$. The latter in general involve groups K of any dimension, whereas spin$_c$ (= spin$_{U(1)}$ in our notation) uses $U(1)$ of dimension 1. The role of K is roughly that of a gauge group, so insisting on the existence of spinors introduces nontrivial gauge symmetry and internal degrees of freedom. In addition, typically, spin$_K$– theories are chiral, left- and right–chiral spinors transforming differently under K. This suggests that there may be a clever way to use fuzzy spaces to get the chiral fermions of the standard model.

There is a simple global appoach to differential geometry on G/H. We introduce this formalism after setting up the preliminaries in Section 2. We follow this up in Section 3 introducing spin– and spin$_K$–structures. Their Dirac and chirality operators are formulated in Section 4. We call this version of the Dirac operator 'Kähler–Dirac operator', as it is similar to the operator with the same name on a complex manifold. There is another equivalent version using projective modules equally useful for fuzzy physics, which we have decided to call the projective Dirac operator. The 'Dirac' operator then refers to either of these two versions. Section 5 takes this up and also establishes its equivalence to the Kähler–Dirac operator. Along the way, the differential geometry of Section 2 is also translated to the language of projective modules. The cut–off versions of these expressions have an important role in fuzzy physics. In Section 5 we also explicitly consider the spheres S^n and $\mathbb{C}P^n$. In particular, for spheres,we compute the curvature and Dirac spectrum for the maximally symmetric metric. Section 6 extends the preceding considerations to gravity on G/H [4]. Finally Section 7 discusses the complex and Kähler structures of coset manifolds.

2 Differential Geometry on G/H

2.1 Preliminaries

G is a simple, simply connected, connected, compact Lie group with Lie algebra $\underline{G}$. $\underline{H}$ is a subalgebra of $\underline{G}$ which by exponentiation generates a compact connected Lie group H.

We think of G concretely as $N \times N$ unitary matrices. The Lie algebra $\underline{H}$ then has a basis $\{T(\alpha)\}$ of hermitean matrices (we follow physics conventions, more correctly $\{iT(\alpha)\}$ span $\underline{H}$), which are trace orthogonal:

Let Ad denote the adjoint representation of G. Then $H \subset G$ leaves $\underline{G/H}$ invariant in this representation:

The above discussion implies the following commutation relations:

$$
\begin{aligned}
[T(\alpha), T(\beta)] &= i c_{\alpha\beta\gamma} T(\gamma) \ , \\
[T(\alpha), S(i)] &= i c_{\alpha i j} S(j) \ , \qquad (2.1) \\
[S(i), S(j)] &= i c_{ij\alpha} T(\alpha) + i c_{ijk} S(k) \ . \qquad (2.2)
\end{aligned}
$$

The structure constants c_{ABC} are real and totally antisymmetric.

We will call c_{ijk} the torsion of the space $\underline{G/H}$. Below we will see that it plays exactly the role of the usual torsion for the canonical covariant derivative on G/H [5]. If $c_{ijk} = 0$, the homogeneous space G/H is said to be 'symmetric' [6]. In that case, $\underline{G}$ admits the involutive automorphism:

2.2 Tensor Fields on G/H

Let W be a fixed vector space with an orthonormal basis $\{e_i\}$ which carries the representation $Ad_{G/H}$ of H, $h \,:\, e_i \to e_j Ad_{G/H}(h)_{ji}$. The vector space $W^{\otimes n} = W \otimes W \otimes \ldots \otimes W$ (n factors) carries the tensor product representation $Ad_{G/H}^{\otimes n} = Ad_{G/H} \otimes Ad_{G/H} \otimes \ldots \otimes Ad_{G/H}$ (n factors). Let $\mathbb{C} \equiv W^{\otimes 0}$ also denote the one–dimensional complex vector space carrying the trivial representation $Ad_{G/H}^{\otimes 0} :\; h \to 1$.

Tensor fields of rank n on G/H can be defined to be equivariant functions on G with values in $W^{\otimes n}$. That means the following: for $n = 0$ we have scalar fields $f^{(0)}$, complex (or $W^{\otimes 0}$) valued functions on G invariant under the right–action of H on G (equivariance):

Let J label the inequivalent irreducible representations of G by unitary matrices $\{D^J(g)\}$; their matrix elements in a convenient orthonormal basis are $D^J_{mn}(g)$. We have that

Henceforth we assume for notational simplicity that the identity representation occurs only once in the restriction of the irreducible representations J of G to H, and so drop the index i_0 from $\xi^J_{mi_0}$. Otherwise a degeneracy index has to be included here and elsewhere.

In the same way, if the representation $h \to D^J(h)$ contains $Ad_{G/H}$, we can choose the basis in the representation space so that the index n in (**??**) transform by $Ad_{G/H}$ if i, j belong to an appropriate index set I:

Continuing in this vein we see that tensor fields of rank n in component form look like $f^{(n)}_{i_1 \ldots i_n}$ and have the expansion

2.3 Covariant Derivative.

Let $\mathcal{T}^{(n)}$ denote the space of tensor fields of rank n, with a typical member $f^{(n)} = \{f^{(n)}_{i_1 \ldots i_n}\}$. $\mathcal{T}^{(0)}$ consists of functions, and it is also an algebra under pointwise multiplication. All $\mathcal{T}^{(n)}$ are $\mathcal{T}^{(0)}$-modules. The covariant derivative ∇ is a map

There is a natural choice for the covariant derivative in our case. We call it hereafter as X. The action of X on functions is:

The torsion of the covariant derivative vanishes only if $[X_i, X_j]\, f^{(0)} = 0$. From the definition and (2.1), we have

Gauge fields will certainly have a central role in further developments. So we briefly indicate what they are here. Let us first consider $U(1)$ gauge fields. The general gauge potential is $A_i = \sum \xi^J_M D^J_{Mi}$, $\xi^J_M \in \mathbb{C}$. It is subject to the reality condition $\overline{A}_i = -A_i$. Then if $f^{(n)}$ has charge e, its covariant derivative is $(\overline{X}_i + eA_i) f^{(n)}_{i_1 \ldots i_n}$, where A_i acts by pointwise multiplication. This definition is compatible with equivariance. We can substitute X_i for $\overline{X}_i$ at the cost of possible torsion.

The gauge covariant derivative for a general gauge group as usual only involves regarding $eA_i(g)$, that is $e\xi^J_M$, to be Lie algebra valued, its action on $f^{(n)}$ in (**??**) is then dictated by the representation content of the latter.

3 Spin– and Spin$_K$–structures

Spinorial fields are essential for physics. We can go about constructing them as follows. The orthogonal group $SO(|G/H|)$ has a double cover $Spin(|G/H|)$. Associated with $SO(|G/H|)$, there is also a Clifford algebra $\mathbb{C}\ell(|G/H|)$ with generators $\gamma_1, \gamma_2, \ldots, \gamma_{|G/H|}$:

A recursive scheme for constructing anticommuting sets of hermitean $\gamma-$matrices goes as follows. We start with a set of $2^{n-1} \times 2^{n-1}$ matrices $\gamma_i,\ i = 1, ..., 2n-1$, satisfying eq.(**??**), and such that $(-i)^{n-1}\gamma_1 ... \gamma_{2n-1} = 1$, e.g. for $n = 2$, the three Pauli matrices. Then a set of $2^n \times 2^n$ matrices $\Gamma_\lambda,\ \lambda = 1, ..., 2n+1$, satisfying eq.(**??**), and such that $(-i)^n \Gamma_1 ... \Gamma_{2n+1} = \mathbb{I}$ is given by

3.1 Spin Manifolds

We say that G/H is a spin manifold if the commutative diagram of Fig.1 exists, arrows being homomorphisms (which need not be onto):

Example 1: $\mathbb{C}P^1 = SO(3)/SO(2) = [Spin(3) = SU(2)]/[Spin(2) = U(1)]$. So $G = SU(2),\ H = U(1) = \{e^{i\sigma_3\theta/2}\}$, σ_A the Pauli matrices. Then $S(i) = \sigma_i,\ i = 1, 2$, and

For a thorough treatment of noncommutative geometry and Dirac operator on S^2, see [7].

Example 2: Similar arguments show that all the spheres $S^N = SO(N+1)/SO(N) = Spin(N+1)/Spin(N)$ are spin. G for S^N is $Spin(N+1)$ while $H = Spin(N)$. $Ad_{G/H}$ is $SO(N)$, the $\mathbb{Z}_2$–quotient of $Spin(N)$. Since $Spin^{\mathbb{C}\ell}(|G/H|)$ is isomorphic to $Spin(N)$, S^N is spin.

Example 3: $\mathbb{C}P^2 = SU(3)/U(2)$. So $G = SU(3),\ H = U(2)$. A basis for the 3–dimensional $SU(3)$– Lie algebra consists of the Gell-Mann matrices λ_A. The $U(2)$ Lie algebra has basis $\lambda_1, \lambda_2, \lambda_3, \lambda_8$, the hypercharge Y being $\frac{1}{\sqrt{3}}\lambda_8$. The $S(i)$ are $\lambda_4, \lambda_5, \lambda_6, \lambda_7$. Under $U(2)$, they transform as (K^+, K^0) or $(-\overline{K}^0, K^-)$ in particle physics notation. That means that $Ad_{G/H} = U(2)$. Regarding $U(2)$ as 2×2 unitary matrices U, we can embed $U(2)$ in $SO(4)$ by the map

Example 4: $G = SU(3),\ H = SO(3)$. With G as 3×3 unitary matrices, H consists of all real orthogonal matrices and corresponds to the spin 1 representation of $SO(3)$. $\underline{G/H}$ is of dimension 5. It carries the spin 2 representation of $SO(3)$, isomorphic (but not equivalent!) to the spin 1 representation. There is no homorphism $SO(3) \to Spin^{\mathbb{C}\ell}(5)$ compatible with Fig.1, so that $SU(3)/SO(3)$ is not spin [3].

Let us show this result in more detail. We can show it by establishing that the 2π–rotation in $SO(3)$ becomes a noncontractible loop in $SO(5)$ under the embedding in Fig.2. Then the inverse image of $SO(3)$ under the homomorphism $Spin^{\mathbb{C}\ell}(5) \to SO(5)$ is $SU(2)$ giving us the result.

Now $SO(3)$ acts on real symmetric traceless 3×3 matrices $T = (T_{ij})$ according to $T \to RTR^T$. This is its spin 2 representation. We can eliminate say T_{33} using $\mathrm{Tr}\, T = 0$, thereby representing it as real transformations on $(T_{11}, T_{12}, T_{13}, T_{22}, T_{23})$. $SO(5)$ consists of real transformations on this five-dimensional vector,so we now have the needed explicit embedding of $SO(3)$ in $SO(5)$. Let

We will now explain the Dirac operators for Spin– and Spin$_K$–manifolds after discussing Spin$_K$-structures.

3.2 Spin$_K$ –Manifolds

K and $\mathcal{H}$ are compact connected Lie groups in what follows. *We say that G/H is a Spin$_K$–manifold if the commutative diagram of Fig.2 exists.*

K and $\mathbb{Z}_2$ on the arrows are to show that they are the kernels of those homomorphisms.

$Spin_{U(1)}$ in our language is what mathematicians call $Spin_c$.

The intersection $Spin^{\mathbb{C}\ell}(|G/H|) \cap K$ clearly contains $\mathbb{Z}_2$. It cannot be larger, for that would mean that the kernel for the slanting arrow exceeds $\mathbb{Z}_2$.
Thus $\mathcal{H} \supset \left[Spin^{\mathbb{C}\ell}(|G/H|) \times K\right]/\mathbb{Z}_2$. Its quotient by K being exactly $SO(|G/H|)$, we conclude that

Let us denote the generators of $\mathbb{Z}_2$ in $Spin^{\mathbb{C}\ell}(|G/H|)$ and K by z_{Spin} and z_K, they square to the respective identities. The inclusion of $Spin^{\mathbb{C}\ell}(|G/H|)$ in Fig.3 is to be understood as follows. The elements of $\mathcal{H}$ are the equivalence classes

As we think of $\mathcal{H}$ as the concrete matrix group obtained by tensoring $Spin^{\mathbb{C}\ell}(|G/H|)$ with a faithful unitary representation of K where z_K is represented by $-\mathbb{I}$

Let us motivate the new requirements in Figures 2 and 3. For a physicist, a spinor changes sign under '2π–rotation'. $\mathcal{H}$ is the group acting on $Spin_K$–spinors. We have required it to contain $Spin^{\mathbb{C}\ell}(|G/H|)$, so we can check this requirement by looking at the action of 2π–rotation $\in Spin^{\mathbb{C}\ell}(|G/H|) \subset \mathcal{H}$. As for asking that $H \to \mathcal{H}$, we can reduce the representation of $\mathcal{H}$ into a direct sum $\oplus\rho$ of irreducible representations ρ of H just as in the discussion of spin structures. The action of the Clifford algebra on $\oplus\rho$ by construction is known. The wave functions of $Spin_K$–spinors are then given by linear spans of representations of G induced by $\oplus\rho$, see (**??**). Later we shall see how the Dirac operator can be defined on these wave functions.

Example 5. $G = SU(3),\ H = U(2),\ G/H = \mathbb{C}P^2$. Here we choose $K = U(1)$. Elements of $U(2)$ can be written as the equivalence classes

Example 6. $G = SU(3),\ H = SO(3)$.

We return to the choices $G = SU(3),\ H = SO(3)$. Choosing $K = U(1)$ is not helpful now, as we lack a suitable homomorphism $H = SO(3) \to \mathcal{H} = Spin^{\mathbb{C}\ell}(5) \otimes_{Z_2} U(1)$. So $SU(3)/SO(3)$ is not even $\mathrm{Spin}_{U(1)}$, a result originally due to Landweber and Stong [3]. A better choice is $K = SU(2)$. Then we can find the homomorphism $H \to \mathcal{H}$ as follows. The image of $Ad_{G/H}$ in $SO(5)$ is an $SO(3)$ subgroup $SO(3)'$. Its inverse image in $Spin^{\mathbb{C}\ell}(5)$ is an $SU(2)$ subgroup $SU(2)'$. Let $\vec{\Sigma}$ and $\vec{T}$ be the angular momentum generators of $SU(2)'$ and K. If $\vec{L}$ are the angular momentum generators of H, the map at the level of Lie algebras is just $\vec{L} \to \vec{\Sigma} + \vec{T}$. Hence $SU(3)/SO(3)$ is $\mathrm{Spin}_{SU(2)}$. More such examples can be found.

3.3 What is $\overline{X}(i)$ now?

We need the extension of the torsion–free connection with components $\overline{X}_i$ to spinors on general $spin_K$–manifolds. The first step in this direction is the extension of X_i.

A spinor field $\psi = (\psi_a)$ on a $Spin_K$–manifold has the expansion

The definition of $\overline{X}_i$ involves the extension of c_{ijk} to spinors so that it can act on the index a. Now, the generators of the $SO(|G/H|)$-Lie algebra are M_{ij} where:

Now

The introduction of gauge fields follows the earlier discussion.

4 The Dirac and Chirality Operators

The massless Dirac operator for the torsion free connection $\overline{X}_i$ is just:

If $|G/H|$ is even, e.g. if G/H is a [co]adjoint orbit, there is also a chirality operator γ anticommuting with D_W:

The subscript 'W' is to indicate that it is the form of the Dirac operator used by the Watamuras [8]. For even $|G/H|$ there is also the unitarily equivalent Dirac operator [9]

5 Projective Modules and their Dirac Operator

5.1 Projective Modules.

In the algebraic approach to vector bundles, their sections are substituted by elements of projective modules ('of finite type') [10]. A projective module is constructed as follows. Let A be an algebra. It can be the commutative algebra $\mathcal{A}$ of C^∞–functions on a manifold M if our interest is in the algebraic description of its vector bundles. But it can also be a noncommutative algebra, in which case there is no evident correspondence with sections of differential geometric vector bundles. Consider $A^N \equiv A \otimes_C \mathbb{C}^N$ with elements $a = (a_1, \dots, a_N),\ a_i \in A$. Let P be an $N \times N$ projector with coefficient in A:

It is very helpful for subsequent developments to have a projective module description of vector bundles. We can find the appropriate projectors by a known method described nicely by Landi [11]. It goes as follows.

Consider for example a rank 1 tensor field and any particular D^J matrix occurring in its expansion, with elements $D^J_{\rho i}$. We have

There is no unique correspondence between projective modules and vector bundles. Thus for each J, we can find a projector and its module. But all such modules are equivalent, since there are elements $\alpha^J = a^J P^J$ and $\alpha^K = a^K P^K$ which naturally correspond for different J and K:

5.2 Differential Geometry

There is much to be said on the differential geometry on projective modules, but for reasons of brevity we limit ourselves to indicating how to extend the definitions of $X(i)$ and $\overline{X}(i)$.

Let us first focus on the tensorial case. Let

The vector fields $\mathcal{L}_A$ are related to the left–invariant vector fields X_A by:

We can define the covariant derivative ∇_ρ on spinors corresponding to X_i in the same way, just changing the index i to a in (**??**), and accordingly changing the choice of J as well.

The canonical torsion c_{ijk} generalizes for tensors to

As for spinors, following (**??**), we define a spinorial torsion which is twice the expression

5.3 The Projective Dirac Operator for Spheres

The equations (**??**) tell us the invertible transformation of a spinor field of § 3.3 to an element of a projective module. So we can transform the Dirac operator D to one acting on this $\mathcal{A}$–module. The result is not illuminating except in special cases like spheres and $\mathbb{C}P^N$, so we take them up first.

a) *Even Spheres*

For $G/H = S^{2n}$, we can choose $G = Spin^{\mathbb{C}\ell}(2n+1) = \{g\}$, $H = Spin^{\mathbb{C}\ell}(2n) = \{h\}$, identifying them with the representations given by γ–matrices, $Spin^{\mathbb{C}\ell}(2n+1)$ and $Spin^{\mathbb{C}\ell}(2n)$. We denote the γ-matrices of H by $\gamma_i,\ i = 1, .., 2n$, and by $\gamma = (-i)^n \gamma_1 \dots \gamma_{2n}$ the additional gamma matrix of G, and call them collectively as $\Gamma_\lambda = (\gamma_i, \gamma),\ \lambda = 1, .., 2n+1$. The generators of H are $\Sigma_{ij} = \frac{1}{4i}[\gamma_i, \gamma_j]$, which together with $\Sigma_{2n+1,i} = \frac{1}{2i}\gamma\gamma_i$ make up the full set of generators $\Sigma_{\mu\nu}$ of G.

The Γ_λ transform as vectors under conjugation by G. That lets us introduce coordinate functions $x = (x_\lambda)$ for S^{2n}, starting from an 'origine' $x^0 = (0, ..., 0, 1)$, as follows:

We let subscript $A = (\mu\nu)$, $\mu > \nu$ stand for either of the multi-indices (ij), (α of Sec.2), or $(2n+1, i)$, (i of Sec.2). For $A = (2n+1, i)$, X_A gives back $X_{2n+1,i} \equiv X_i$ of Sec.2, which is now torsionless, G/H being symmetric.

Since Γ_{2n+1} commutes with Σ_{ij}, D_W can be written as:

We choose J to correspond to the preceding Clifford representation to fix the spinorial projective module. We now show that on this module the above Dirac operators have the beautiful forms

In fact, if we apply (**??**), since by (**??**) $\mathcal{J}_A(D^{\mathbb{C}\ell})^\dagger = 0$, we can see that

When acting on functions on S^{2n}, we can use our coordinates to express the right-invariant vector fields in the form

To determine the spectrum and eigenspinors of the Dirac operator we need to be more explicit about the group $Spin(2n+1)$. It has rank n, and IRR's that can be labeled by the components of the highest weight $(m_1, ..., m_n)$, with the m_i's all integers or all half integers, and $m_1 \geq m_2 \geq ... \geq m_n \geq 0$. The Clifford representation $Spin^{\mathbb{C}\ell}$ has highest weight $(\frac{1}{2}, .., \frac{1}{2})$, dimension 2^n and quadratic Casimir operator $C_2(\mathbb{C}\ell) \equiv C_2(Spin^{\mathbb{C}\ell}) = \frac{1}{2}\Sigma_{\mu\nu}\Sigma_{\mu\nu} = \frac{1}{4}n(2n+1)$. We indicate by L an IRR associated with the set I_0 of §2.2; it has highest weight $(l, 0, .., 0)$, where l is an integer, and dimension and quadratic Casimir operator

b) *Odd Spheres*

An odd sphere $S^{2n-1} = SO(2n)/SO(2n-1)$ differs from an even sphere S^{2n} in important details. The Clifford algebra $\mathbb{C}\ell(2n-1)$ has two inequivalent 2^{n-1}-dimensional representations, with $(-i)^{n-1}\gamma_1...\gamma_{2n-1} = \mathbb{I}$ and $(-i)^{n-1}\tilde{\gamma}_1...\tilde{\gamma}_{2n-1} = -\mathbb{I}$; we may take $\tilde{\gamma}_i = -\gamma_i$, which makes clear that they give a single IRR's of $Spin(2n-1)$, with generators $\frac{1}{4i}[\gamma_i, \gamma_j]$. They do give however two inequivalent IRR's of $Spin(2n)$, with generators $(\frac{1}{4i}[\gamma_i, \gamma_j], -\frac{1}{2}\gamma_i)$ and $(\frac{1}{4i}[\gamma_i, \gamma_j], \frac{1}{2}\gamma_i)$, let us label them $\mathbb{C}\ell^+$ and $\mathbb{C}\ell^-$.

For covariance it is better to put these two representations together and work with the 2^n-dimensional Γ_μ, $\mu = 1, ..., 2n$, built from the γ_i-s as indicated in (**??**); that particular construction gives

Spinors carry the direct sum of these two IRR's on their index, and we can use either of the Dirac operators

There is no chirality in odd dimensions, but Γ_{2n} plays a role in space(time)-reflection, and can be used to give Dirac operators equivalent to $D_W^{(1,2)}$ [9]:

Proceeding as we did for even spheres, with $\mathcal{L}_{\mu\nu}$ as in eq.(**??**) the Dirac operators can be rewritten in the form

With all this information, the analogues of (**??**),(**??**) give for the eigenvalues

5.4 The Projective Dirac Operators on $\mathbb{C}P^N$

For reasons of brevity, we focus on $\mathbb{C}P^2$, a case we have already treated in [12]. $\mathbb{C}P^2$ is $SU(3)/U(2)$. If λ_α are the Gell-Mann matrices, it is the orbit of λ_8 under $SU(3)$:

If we can achieve a covariant–looking form for D and D_W looking like (**??**), (**??**), we can find covariant $\mathcal{D}$ and $\mathcal{D}_W$. Towards this end we introduce the Clifford algebra with eight generators γ_A. They can be transformed by the adjoint representation of $SU(3)$ without disturbing their anticommutators:

Consider the action $\gamma_A \to [\Sigma_8, \gamma_A]$ of Σ_8 on γ_A. For this action, the eigenvalues of Σ_8 are $\pm\frac{\sqrt{3}}{2}$ and 0. The 0 eigenvalues are for γ_A with $A = 1, 2, 3, 8$, thus:

$$\begin{aligned} [\Sigma_8, [\Sigma_8, \gamma_A]] &= 0 \quad \text{if} \quad A = 1, 2, 3, 8 \ , \\ &= \frac{3}{4}\gamma_A \quad \text{if} \quad A = 4, 5, 6, 7 \ . \end{aligned} \tag{5.1}$$

This lets us write the Dirac operator in 'covariant' form

For the projective module, for the representation D^J, we have the one given by Σ_A. It is $2^{8/2} = 16$- dimensional. The transform $\mathcal{D}_W$ of D_W onto this module is immediate:

6 On Riemannian Structure and Gravity.

An inverse metric (η^{ij}) is a symmetric nondegenerate field, which defines a map $\mathcal{T}^{(1)} \otimes \mathcal{T}^{(1)} \to \mathcal{T}^{(0)}$ via $f \otimes f' \to \eta^{ij} f_i f'_j$. As the f's transform by $Ad_{G/H}$ under $g \to gh$, (η^{ij}) transform by the product $Ad^{-1}_{G/H} \otimes Ad^{-1}_{G/H}$ of its contragradient representation. Or, the metric (η_{ij}) itself transforms by $Ad_{G/H} \otimes Ad_{G/H}$.

A particular metric is $(\hat{\delta}_{ij})$, where $\hat{\delta}_{ij}(g)$ is δ_{ij} (δ= Kronecker δ). The torsion–free covariant derivative compatible with $\hat{\delta}$ is $\overline{X}$:

A more general H-invariant metric η can be defined as follows. Let us decompose $\underline{G/H}$ into irreducible subspaces under $Ad_{G/H}$ and let $\{S^{(\sigma)}_m\}$ be a basis for the unitary irreducible representation σ such that

The general covariant differential ∇ can be defined in the usual way:

Next, introduce $|G/H|$-beins or soldering forms e^a_i such that

The spin connection is defined by:

The covariant derivative on spinors ψ is given by

The Dirac operator in the presence of a gravity field (η_{ij}) is thus:

All this stuff is very natural. It remains to transport it to projective modules. In the module picture η_{ij} gets transformed to

The projective module analogue of $\overline{X}_i$ is the $\overline{\nabla}_\rho$ defined in section 5.2. Adding the action of

The action of ∇_ρ on spinorial modules follows from (**??**). We let $\mathcal{J}_\rho$ be the total angular momentum for the representation J_S chosen for spinors, and

$$\begin{aligned}
\mathcal{C}^{(S)}_{\rho\lambda\sigma} &= D^{J_S}_{\lambda b}(c_{ijk}\gamma_j\gamma_k)_{ba}(D^{J_S})^\dagger_{a\sigma}(D^{J_T})^\dagger_{i\rho} \ , \\
\Omega_{\rho\lambda\sigma} &= D^{J_S}_{\lambda b}((\omega_i)_{jk}\gamma_j\gamma_k)_{ba}(D^{J_S})^\dagger_{a\sigma}(D^{J_T})^\dagger_{i\rho} \ , \\
\chi_\sigma &= \psi_a(D^{J_S})^\dagger_{a\sigma} \ .
\end{aligned} \tag{6.1}$$

Then, as can easily be shown from (**??**) above,

7 Complex Structures and Kähler Manifolds

In favourable circumstances, we can push this program ahead and define more refined ideas like complex and Kähler structures on tensors $\mathcal{T}^{(n)}$ and on their projective modules. We indicate how to treat them briefly.

We consider adjoint orbits only for G/H. Thus let $\underline{k}$ be a fixed element of $\underline{G}$ from the Cartan subalgebra $C(\underline{G})$, and H its stability group:

Consider the eigenvalue equation

By (2.1), $(\underline{G/H})_c$ is invariant under the adjoint action of $\underline{k}$. Also, as $Ad_{\underline{G/H}}$ is a real, orthogonal representation, the eigenvalues λ_a are real, while of course the $\overline{S}(i)$ are hermitean. So the adjoint of (**??**) shows that $E_a^\dagger$ corresponds to the eigenvalue $-\lambda_a$, and that each positive eigenvalue is paired with a negative one. The eigenvalues λ_a may be degenerate.

We choose $E_a,\ a = 1, ..., \frac{1}{2}(|G| - |H|)$, to be solutions of (**??**) with $\lambda_a > 0$, $E_{-a} = E_a^\dagger$, and the normalization

Let $(\underline{G/H})_c^\pm$ denote the span of the eigenvectors $E_{\pm|a|}$ (where note that $|a| > 0$). The subspaces $(\underline{G/H})_c^\pm$ are of precisely the same dimension and

This complex structure is Kähler. To show it, let us introduce the Maurer-Cartan forms θ^A, defined by $g^{-1}dg = i\Sigma_A\theta^A$, or, setting $c = 1$ in (**??**), by

Consider the particular Maurer-Cartan form

The Kähler metric η on vector fields $(X_a,\, X_b)$ is given by

Finally, we shall show that the Kähler metric on G/H can be derived from a Kähler potential Φ_ζ. It is a function on G/H and depends on a parameter ζ. It has the property

The construction of Φ_ζ involves the member of a specific class of unitary representations $\Sigma_K : g \to \Sigma_K(g)$ of G. Let σ_K be the associated representation of $\underline{G}$. Any such representation contains a normalized highest weight vector $|K>$ with eigenvalue K for $\sigma_K(\underline{k})$, which is annihilated by the orthogonal complement of $\underline{k}$ in $\underline{H}$ and the positive roots E_a:

$$
\begin{aligned}
&\text{a)} \quad \sigma_K(\underline{k})|K >= K|K > \quad , \ K > 0 \ , \\
&\text{b)} \quad \sigma_K(T(\alpha))|K >= 0 \quad \text{if } \ \mathrm{Tr}\, T(\alpha)\underline{k} = 0 \ , \\
&\text{c)} \quad \sigma_K(E_a)|K >= 0 \quad \text{for } \forall\ a > 0 \ .
\end{aligned} \tag{7.1}
$$

A representation of G fulfilling a) and b) always exists: it is induced from the unitary one-dimensional representation of H given by a) and b):

Let us fix an orthonormal basis $\{e_1, e_2, \ldots, e_M\}$ in the representation space of dimension M (say) of $\Sigma_K(G)$. Choose a vector $|\zeta) = \sum_{i=1}^{M} \zeta_i e_i$, $\zeta_i \in \mathbb{C}$, $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_M)$, so that $(\zeta|\Sigma_K(g)|K > \neq 0$ when g belongs to some open set $\mathcal{O}$. Such a $|\zeta)$ exists since $(\zeta|\Sigma_K(g)|K >= 1$ for $|\zeta) = \Sigma_K(g)|K >$. Further choose $\mathcal{O}$ so that it is invariant under H-action. That is always possible since for $(\zeta|\Sigma_K(g)|K >$ changes only by a phase under this action by (**??**).

Now the function ω_ζ defined by

If $g \in \mathcal{O}$, the Kähler potential is given by the formula

In another open set $\mathcal{O}' \subset G$, we may have to work with the Kähler potential Φ_η. Then if $\mathcal{O} \cap \mathcal{O}' \neq \emptyset$, the two potentials on $\mathcal{O} \cap \mathcal{O}'$ are related by

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